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CONVECTIVE HEAT TRANSFER FOR STEADY LAMINAR FLOW BETWEEN TWO CONFOCAL ELLIPTIC PIPES WITH LONGITUDINAL UNIFORM WALL TEMPERATURE GRADIENT

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Abstract—Convective heat transfer for steady laminar flow between two confocal elliptic pipes with longitudinal uniform wall temperature gradient under various heating conditions is presented in analytical closed form utilizing the exact solutions of the Navier–Stokes and energy equations. It is shown that one characteristic number, the product of the dimensionless longitudinal uniform wall temperature gradient and Peclet number, effects the problem. The values of Nusselt numbers for several values of ellipticity and core size are obtained.

NOMENCLATURE

A, B, semi-axes of outer periphery;
A_{01}, B_{01}, D_{01}, functions defined by equations (A.2);
A_ω, B_ω, semi-axes of inner periphery;
a_1, a factor defined by equation (3.12);
C, c, dimensional and dimensionless temperature gradients along the pipe;
D, hydraulic diameter defined by equation (5.8);
DENO, DENI, functions defined by equation (5.19);
$(DE)_1, (DE)_\omega, DE$ factors defined by equations (5.14) and (5.15);
dU, element of heat flux in the direction of ξ across an element of ξ = const. elliptic cylinder;
E, e, dimensional and dimensionless excess temperatures;
E_{in}, excess temperature of inner pipe;
E_m, mixed mean excess temperature;
E_1, E_ω, elliptic integrals of the second kind for outer and inner walls, respectively;
E_0, E_2, E_4, factors defined by equations (3.7);
F, a characteristic heat flux defined in equations (3.5);
G, a dimensionless factor defined in equations (3.7);
$g_0, f_0, f_2, e_0, e_2, e_4$, coefficients defined in equation (3.9);
H, h, dimensional and dimensionless heat generation densities;
I_{00}, a function defined by equation (2.13);
I_{01}, a function defined by equation (A.1);
J, a function defined by equation (5.16);
K_0, K_1, K_2, K_3, K_4, functions defined by equations (6.2);
k, thermal conductivity;
L_0, mean of the semi-axes of outer periphery;
m, coefficient of ellipticity defined by equation (2.6);
Nu_o, Nu_i, Nusselt numbers on outer and inner walls;
Pe, Peclet number;
P, p, dimensional and dimensionless pressures;
Pr, Prandtl number, ν/α,
P_o, P_i, circumferences of outer and inner peripheries;
Q, rate of mass flow;
Q_S, rate of mass flow for a simple elliptic pipe
Re, Reynolds number;
S, dS, full and elemental cross sectional areas of elliptic annular pipe;

T,	temperature;
T_o, T_i,	outer and inner wall temperatures;
T_m,	mixed mean temperature;
t,	alternate elliptic coordinate, $t = \log \xi$;
U_i, U_o,	heat fluxes from inner and outer walls;
U_0, V_0, W_0, Y_0, Z_0,	functions defined by equations (A.4);
U_2, V_2, Z_2,	functions defined by equations (A.6);
U_4, Z_4,	functions defined by equations (A.8);
u,	a function defined in equations (2.11);
V,	velocity vector;
W, w,	dimensional and dimensionless velocities;
w_o, w_2,	functions defined by equations (2.11);
X, Y, x, y,	dimensional and dimensionless transversal coordinates;
x_2,	a function defined by equation (2.8);
Y, J_0, J_2, J_4,	integrals defined in equation (5.17);
Z, z,	dimensional and dimensionless longitudinal coordinates.

Greek symbols

α,	thermal diffusivity;
β, β_i, β_0,	factors defined by equations (4.4) and (4.5);
λ, λ_o,	factors defined by equations (4.7) and (4.8);
μ,	ratio of λ to λ_o, λ/λ_o;
μu,	coefficient of viscosity;
v,	kinematic viscosity;
ξ, η,	dimensionless elliptic coordinates;
ρ,	density;
ξ_o,	elliptic coordinate of the point where velocity is maximum;
ω,	elliptic coordinate of inner periphery.

1. INTRODUCTION

CONVECTIVE heat transfer in steady laminar flow for various geometries have been extensively covered in existing literature. Kays [1] has discussed the majority of the more important cases and has given the heat transfer results. However, the one important case of heat transfer between two confocal elliptic pipes with various wall heating conditions is not included in these studies. Besides the scientific interest of this case, the design of a heat exchanger in a narrow space may require information on convective heat transfer in annular elliptic pipes. Additionally, the limiting case of an elliptic pipe with a flat core, heated or cooled independently on its internal and external walls, may also find useful engineering applications.

The problem analyzed here is the convective heat transfer in a steady laminar flow between two pipes having confocal elliptic cross sections with walls heated or cooled independently and subjected to uniform longitudinal wall temperature gradients. In obtaining the velocity and temperature fields, a uniform heat generation is included. However, for the derivation of heat-transfer coefficients on the inner and outer walls, the heat generation is omitted. The velocity and temperature distributions are obtained as exact solutions of the Navier–Stokes and energy equations, respectively; and are presented in closed forms.

It is shown that the temperature distribution, equation (3.9), and heat transfer, equation (4.11), through the walls are affected by one parameter which is the product of the dimensionless applied uniform longitudinall wall temperature gradient and the Peclet—*Pe*—number in addition to the geometric parameters of the cross section of the walls. The Nusselt—*Nu*—numbers for the inner and outer walls are obtained explicitly and are calculated for thirteen different values of the flatness ratio of the outer pipe and for five typical wall heating combinations.* These results are given in tabular form. For four values of the ellipticity and for the special case of ellipticity equal to the core size, the results are also given graphically to illustrate the heat-transfer characteristics of the problem. The results for the special case of circular annular pipes are compared with the data as obtained by Lundberg *et al.* [2], and agreement is found within the limits of the number of significant digits as given in this reference.

2. VISCOUS FLOW IN ANNULAR ELLIPTIC PIPES

The velocity W for steady laminar flow of a viscous fluid flowing in a straight conduit under a uniform longitudinal pressure gradient $\partial P/\partial Z$ satisfies the equation

$$\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}\right)W=\frac{1}{\mu u}\frac{\partial P}{\partial Z} \tag{2.1}$$

where Z is the longitudinal coordinate in the direction of flow, X and Y are the rectangular transverse coordinates, P is the pressure and μu is the dynamic coefficient of viscosity of the fluid. Let the semi-major and semi-minor axes of the cross section of the outer elliptic pipe be A and B, respectively. Denoting the mean of the semi-axes of the elliptic section of the outer pipe by

$$L = \tfrac{1}{2}(A+B) \tag{2.2}$$

*Due to the page restriction only three tables for ellipticity equal to 0·0, 0·7 and 0·8 are included. The remaining tables for ellipticity equal to 0·1, 0·2, 0·3, 0·4, 0·5, 0·6, 0·9, 0·94, 0·96 and 0·98 may be obtained from either author upon request.

and using the transformations

$$X = Lx, \quad Y = Ly, \quad Z = Lz, \quad W = \frac{\nu}{L}w,$$

$$\frac{\partial P}{\partial Z} = \frac{\mu u \nu}{L^3}\frac{\partial p}{\partial z} \tag{2.3}$$

where ν is the kinematic viscosity and the lower case letters are the dimensionless forms of their upper case counterparts, (2.1) may be nondimensionalized. The right hand side of the new differential equation will include only one parameter, the Reynolds number.

A simple definition of Reynolds—Re—number for a pipe with non-circular cross section is described in [3]. According to this, the Reynolds number for an elliptic annular pipe of mean outer radius L_0 and under a pressure gradient $\partial P/\partial Z$ is defined by the Reynolds number in a simple circular pipe of radius L_0 under the same pressure gradient. This definition is used in this work.

The elliptic boundaries require the use of elliptic coordinates. Taking the origin of the coordinates x and y at the centerline of the pipe, the dimensionless elliptic coordinates, ξ and η, can be obtained from equations:

$$x = \xi\left(1+\frac{m^2}{\xi^2}\right)\cos\eta, \quad y = \xi\left(1-\frac{m^2}{\xi^2}\right)\sin\eta \tag{2.4}$$

where m, a positive constant between zero and one, depends upon the size of the semi-major and semi-minor axes of the outer pipe. Letting the periphery of the inner pipe be an ellipse, confocal with that of the outer pipe, having an elliptic coordinate ω and semi-major and semi-minor axes A_ω and B_ω, respectively, we have

$$\begin{aligned} &\xi = 1 \quad \text{on the outer periphery} \\ &\xi = \omega \quad \text{on the inner periphery} \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} A &= (1+m^2)L, \quad B = (1-m^2)L \\ A_\omega &= \omega\left(1+\frac{m^2}{\omega^2}\right)L, \quad B_\omega = \omega\left(1-\frac{m^2}{\omega^2}\right)L \\ m &= \left[\left(1-\frac{B}{A}\right)\Big/\left(1+\frac{B}{A}\right)\right]^{1/2} \end{aligned} \tag{2.6}$$

Using an alternate system as, $t = \log\xi$ and η, the differential equation for w changes to

$$\left(\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial \eta^2}\right)w = -4Re(x_2 - 2m^2\cos 2\eta) \tag{2.7}$$

where

$$x_2 = \xi^2 + \frac{m^4}{\xi^2} \tag{2.8}$$

together with the boundary conditions

$$\begin{aligned} &w = 0 \quad \text{at} \quad \xi = 1 \quad \text{and} \quad \xi = \omega \\ &\text{or} \quad \text{at} \quad t = 0 \quad \text{and} \quad t = \log\omega. \end{aligned} \tag{2.9}$$

The solution is obtained as

$$w = Re(w_o - w_2\cos 2\eta) \tag{2.10}$$

where

$$\begin{aligned} w_0 &= (1-\xi^2)\left(1-\frac{m^4}{\xi^2}\right) + u\log\xi \\ w_2 &= \frac{2m^2}{1+\omega^2}(1-\xi^2)\left(1-\frac{\omega^2}{\xi^2}\right) \\ u &= -(1-\omega^2)\left(1-\frac{m^4}{\omega^2}\right)\Big/\log\omega. \end{aligned} \tag{2.11}$$

One important fact of this field concerns the locations where the velocity is a maximum. The velocity, w, reaches maximum values at two symmetric points on the minor axis of the cross-section, one above and one below the origin, having the elliptic coordinate ξ_o whose square is

$$\xi_o^2 = \frac{u}{4}\left[\frac{(1+\omega^2)}{1+2m^2+\omega^2}\right] \times\left[1+\sqrt{\left(1+16\frac{m^2}{u^2}\frac{(1+\omega^2)(1+m^2)+2(m^2+\omega^2)}{(1+\omega^2)^2}\right)}\right]. \tag{2.12}$$

The rate of mass flow through the annulus of the elliptic pipes is

$$Q = 2\pi(\mu u)L\,Re\,I_{00} \tag{2.13}$$

where

$$I_{00} = \tfrac{1}{4}(1-\omega^4)\left(1+\frac{m^8}{\omega^4}\right) - 2m^4\frac{(1-\omega^2)}{(1+\omega^2)} - \tfrac{1}{4}(1-\omega^2)\left(1-\frac{m^4}{\omega^2}\right)u. \tag{2.14}$$

For the case of circular concentric pipes, $m = 0$, Q reduces to that of [4].

The rate of mass flow expression, (2.13), by using the expression of the rate of mass flow, Q_S, [4] for a simple elliptic pipe of semi-axes A and B can also be written as

$$\frac{Q}{Q_S} = 4\frac{(1+m^4)}{(1-m^4)^3}I_{00} \tag{2.15}$$

which represents the reduction of flow due to the presence of the inner pipe.

3. TEMPERATURE DISTRIBUTION

Consider an elliptic annular pipe subjected to two independent axially uniform heat fluxes through its inner and outer walls. Along the length of the pipe where the velocity and temperature distributions are fully developed, the temperature distribution must have the functional form

$$T = CZ + E(X, Y) \tag{3.1}$$

where C is the constant temperature gradient along the pipe. Furthermore, presume that each of the inner and outer wall temperature distributions are peripherally uniform. Then, the second term $E(X, Y)$ of equation (3.1) represents an excess temperature for which one of the end values can be taken as zero without loss of generality. Selecting the outer value as zero and denoting the inner value by E_{in}, the boundary conditions of $E(X, Y)$ in terms of the elliptic coordinate ξ satisfy

$$E(1) = 0, \quad E(\omega) = E_{in} \tag{3.2}$$

and the outer and inner wall temperatures, T_o and T_i, at any station Z become

$$T_o = CZ, \quad T_i = T_o + E_{in}. \tag{3.3}$$

Including a uniform heat generation density, H, and neglecting the viscous dissipation, the temperature distribution satisfies the steady state energy equation

$$\mathbf{V}.\operatorname{grad} T = \alpha\left(\nabla^2 T + \frac{H}{k}\right) \tag{3.4}$$

where $\mathbf{V}$ is the velocity vector, α is the thermal diffusivity, k is the thermal conductivity. Using a characteristic heat flux, F, such as the averaged heat flux from the outer and inner walls, and the transformations

$$H = \frac{F}{L}h, \quad C = \frac{F}{k}c, \quad E = \frac{LF}{k}e \tag{3.5}$$

the energy equation can be non-dimensionalized. The result in the elliptic coordinates is

$$\left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial \eta^2}\right)e = -h(x_2 - 2m^2\cos 2\eta) + G(E_o - E_2\cos 2\eta + E_4\cos 4\eta) \tag{3.6}$$

where

$$G = cPe$$
$$E_o = x_2 w_o + m^2 w_2$$
$$E_2 = 2m^2 w_o + x_2 w_2$$
$$E_4 = m^2 w_2 \tag{3.7}$$

and Pe represents the Peclet number. The boundary conditions for excess temperature in terms of the elliptic coordinate ξ are

$$e(1) = 0, \quad e(\omega) = e_{in} \tag{3.8}$$

where e_{in} is the dimensionless excess temperature on the inner wall.

The equation of e, (3.6), shows that the excess temperature distribution, besides the heat generation density h, is affected only by one dimensionless parameter G, the product of the dimensionless uniform longitudinal wall temperature gradient and the Peclet number.

The solution of (3.6) is obtained as

$$e = e_{in}g_o + h(f_o + f_2\cos 2\eta) + G(e_o + e_2\cos 2\eta + e_4\cos 4\eta) \tag{3.9}$$

where

$$g_0 = \frac{\log \xi}{\log \omega}, \quad f_0 = \tfrac{1}{4}w_o, \quad f_2 = -\tfrac{1}{4}w_2 \tag{3.10}$$

and

$$e_0 = -\tfrac{1}{4}(1+m^4)w_0 + \tfrac{1}{4}m^2 w_2 + \tfrac{1}{4}u\left(1-\frac{\omega^2}{\xi^2}\right)\left(\xi^2 - \frac{m^4}{\omega^2}\right)\log\xi + \frac{1}{16}\left[(1-\xi^4)\left(1-\frac{m^8}{\xi^4}\right) + u(1+\omega^2)\left(1+\frac{m^4}{\omega^2}\right)\log\xi\right] + \tfrac{1}{4}u\left[(1-\xi^2)\left(1+\frac{m^4}{\xi^2}\right) + ua_1\log\xi\right] \tag{3.11}$$

where

$$a_1 = \left(1+\frac{m^4}{\omega^2}\right)\Big/\left(1-\frac{m^4}{\omega^2}\right) \tag{3.12}$$

$$e_2 = -\frac{1}{6}\frac{m^2}{(1+\omega^2)}\left[(1-\xi^4)\left(1-\frac{m^4\omega^2}{\xi^4}\right) + u(1+\omega^2)\log\xi\right] + \frac{2}{3}\frac{m^2}{(1+\omega^2)}\left[(1-\xi^2)\left(1-\frac{m^4\omega^2}{\xi^2}\right) + u(1+\omega^2)\log\xi - \frac{\omega^4}{(1+\omega^2)}\left(1-\frac{m^4}{\omega^4}\right)\left(\xi^2 - \frac{1}{\xi^2}\right)\right] \tag{3.13}$$

$$e_4 = -\frac{1}{6}\frac{m^4}{(1+\omega^2)}(1-\xi^2)\left(1-\frac{\omega^2}{\xi^2}\right) + \frac{1}{24}\frac{m^4}{(1+\omega^4)}(1-\xi^4)\left(1-\frac{\omega^4}{\xi^4}\right). \tag{3.14}$$

For the simple case of a circular pipe, $m = 0$ and $\omega = 0$, the elliptic coordinate ξ becomes the polar radial coordinate r and equation (3.9) reduces to that of [5].

4. HEAT FLUXES THROUGH THE WALLS

The element of heat flux, dU, measured in the positive direction of ξ through an elemental area of ξ = const. cylindrical surface is

$$dU = -LF\xi\frac{\partial e}{\partial \xi}d\eta. \tag{4.1}$$

The heat gain rates U_i and U_o, per unit length of inner and outer pipes respectively, which are taken to be positive when heat flows into the fluid, are expressed as

$$\text{From the inner wall: } U_i = -LF\omega \int_0^{2\pi} \left(\frac{\partial e}{\partial \xi}\right)_{\xi=\omega} d\eta$$

$$\text{From the outer wall: } U_o = LF \int_0^{2\pi} \left(\frac{\partial e}{\partial \xi}\right)_{\xi=1} d\eta. \quad (4.2)$$

Neglecting the heat generation and substituting e from equation (3.9) one obtains

$$U_i = -2\pi LFG \frac{\beta + e_o'(\omega)\omega \log \omega}{\log \omega}$$
$$U_o = 2\pi LFG \frac{\beta + e_o'(1)\log \omega}{\log \omega} \quad (4.3)$$

with $e_o'(\omega)$ and $e_o'(1)$ being the derivatives of e with respect to ξ evaluated at $\xi = \omega$ and $\xi = 1$ and

$$\beta = \frac{E_{\text{in}}}{LC\,Pe} \quad (4.4)$$

which is an alternate definition for the dimensionless inner wall excess temperature.

Two special values of β are as follows:

For insulated outer wall ($U_o = 0$):

$$\beta = \beta_i = -e_o'(1)\log \omega$$

For insulated inner wall ($U_i = 0$):

$$\beta = \beta_o = -e_o'(\omega)\omega \log \omega \quad (4.5)$$

the difference between these being

$$\beta_i - \beta_o = -I_{00} \log \omega. \quad (4.6)$$

The ratio, λ, of the heat gains from the outer wall to that of from both walls per unit length of pipe is obtained as

$$\lambda = \frac{U_o}{U_o + U_i} = \frac{\beta - \beta_i}{\beta_o - \beta_i} \quad (4.7)$$

depending only on the dimensionless inner wall temperature β.

For the special case of equal wall temperatures, $T_o = T_i (\beta = 0)$ the ratio λ is

$$\lambda_o = \frac{e_o'(1)}{I_{00}} \quad (4.8)$$

where

$$e_o'(1) = \tfrac{1}{4}(1-m^8) - \frac{(1-\omega^2)}{(1+\omega^2)} m^4 - \tfrac{1}{4}u(1+m^4)$$
$$- \tfrac{3}{16}u(1+\omega^2)\left(1 + \frac{m^4}{\omega^2}\right) + \tfrac{1}{4}u^2 a_1. \quad (4.9)$$

Noting from the equation (2.13) that $I_{00} > 0$, the sign of $e_o'(1)$ must be the same as that of λ_o.

Introducing the ratio, $\mu = \lambda/\lambda_o$, the alternate dimensionless inner wall excess temperature, β, can also be expressed as

$$\beta = -(1-\mu)e_o'(1)\log \omega. \quad (4.10)$$

After substituting this into equation (4.3), the inner and outer heat fluxes are reduced to

$$U_o = 2\pi LFG\mu e_o'(1)$$
$$U_i = 2\pi LFG[I_{00} - \mu e_o'(1)]. \quad (4.11)$$

For three special cases the values of λ, β and μ are:

For insulated outer wall ($U_o = 0$): $\lambda = 0$, $\beta = \beta_i$, $\mu = 0$;

For equal wall temperatures ($T_o = T_i$): $\lambda = \lambda_o$, $\beta = 0$, $\mu = 1$;

For insulated inner wall ($U_i = 0$): $\lambda = 1$, $\beta = \beta_o$, $\mu = \mu_o = 1/\lambda_o$. (4.12)

For the cases of equidirectional heat fluxes through both walls μ must fall into the following ranges:

$$\text{For } \lambda > 1 \quad \mu > \mu_o \quad -1 < \frac{U_i}{U_o} < 0$$

$$\text{For } \lambda < 0 \quad \mu = -n^2\mu_0 \quad \frac{U_i}{U_o} < -1 \quad (4.13)$$

where n^2 is any positive factor.

Now it is clear that any possible combinations of the heat fluxes through the walls can be represented by either of the parameters λ or μ as illustrated in Fig. 1.

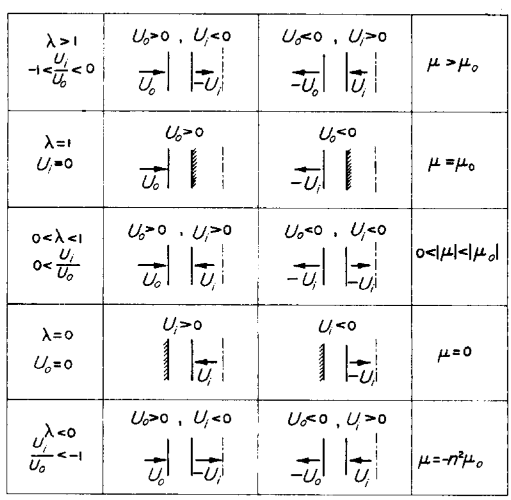

FIG. 1. All possible heating combinations for inner and outer walls.

5. CONVECTIVE HEAT-TRANSFER COEFFICIENTS

The mixed mean or bulk temperature, T_m, at any station Z is defined by

$$T_m = \frac{\rho}{Q}\int_S WT\,\mathrm{d}S$$

where S and $\mathrm{d}S$ are full and elemental cross sectional areas respectively. A mixed mean excess temperature, E_m, can be similarly defined as

$$E_m = T_m - T_o = \frac{\rho}{Q}\int_S WE\,\mathrm{d}S. \qquad (5.2)$$

Substituting for Q from (2.13), this becomes

$$E_m = -\frac{FL}{k}\frac{J}{I_{00}} \qquad (5.3)$$

where

$$J = -\frac{1}{2\pi Re}\int_0^{2\pi}\int_\omega^1 \xi\left(1+\frac{m^4}{\xi^4}-2\frac{m^2}{\xi^2}\cos 2\eta\right)we\,\mathrm{d}\xi\,\mathrm{d}\eta. \qquad (5.4)$$

Noting that, for noncircular cross sections, peripherally uniform temperature distributions do not correspond to uniform heat flux distributions around the peripheries, the mean convective heat-transfer coefficients h_o and h_i for the outer and inner walls may be defined from the equations

$$U_o = (T_o - T_m)P_o h_o, \quad U_i = (T_i - T_m)P_i h_i \qquad (5.5)$$

where P_o and P_i are the circumferences of the outer and inner elliptic pipes. Employing equations (3.3) and (5.2), these may be re-expressed as

$$U_o = -E_m P_o h_o, \quad U_i = -(E_m - E_{in})P_i h_i. \qquad (5.6)$$

Dimensionless mean convective heat-transfer coefficients (the Nusselt numbers) for the outer and inner walls based on the hydraulic diameter of the pipe are then

$$Nu_o = \frac{D}{k}h_o, \quad Nu_i = \frac{D}{k}h_i \qquad (5.7)$$

where

$$D = 4\pi\left[\frac{AB - A_\omega B_\omega}{P_o + P_i}\right] \qquad (5.8)$$

represents the hydraulic diameter of the elliptic annular pipe.

The circumferences of the outer and inner pipes are

$$P_o = 4AE_1, \quad P_i = 4A_\omega E_\omega \qquad (5.9)$$

where E_1 and E_ω are complete elliptic integrals of the second kind which are functions of the eccentricities of the outer and inner pipes,

$$\zeta(1) = \frac{2m}{(1+m^2)}, \quad \zeta(\omega) = \frac{2}{\left[1+\frac{m^2}{\omega^2}\right]}\frac{m}{\omega}, \qquad (5.10)$$

respectively. Substituting equations (2.6) and (5.9) into equation (5.8)

$$D = \frac{\pi(1-\omega^2)\left(1+\frac{m^4}{\omega^2}\right)}{(1+m^2)E_1 + \left(1+\frac{m^2}{\omega^2}\right)\omega E_\omega}L. \qquad (5.11)$$

Also by using equations (3.7), (4.4) and (5.3) the factor in the second equation of (5.6) can be changed to

$$E_m - E_{in} = -\frac{FL}{k}\left(\frac{J}{I_{00}} + G\beta\right). \qquad (5.12)$$

Now substitution of equations (4.11), (5.3), (5.9), (5.11), and (5.12) into equation (5.7) yields the Nusselt numbers

$$Nu_o = (DE)_1\left[\frac{I_{00}}{J/G}\right]\mu e_o'(1),$$

$$Nu_i = (DE)_\omega \frac{I_{00} - \mu e_o'(1)}{\beta I_{00} + \frac{J}{G}} I_{00} \qquad (5.13)$$

where

$$(DE)_1 = \frac{\pi^2(DE)}{2(1+m^2)E_1}, \quad (DE)_\omega = \frac{\pi^2(DE)}{2\left(1+\frac{m^2}{\omega^2}\right)\omega E_\omega} \qquad (5.14)$$

and

$$DE = \frac{(1-\omega^2)\left(1+\frac{m^4}{\omega^2}\right)}{(1+m^2)E_1 + \left(1+\frac{m^2}{\omega^2}\right)\omega E_\omega}. \qquad (5.15)$$

The integral J in the case of no heat generation density is calculated as

$$\frac{J}{G} = \beta Y + J_0 + J_2 + J_4 \qquad (5.16)$$

where

$$Y = -\int_\omega^1 E_o g_o \frac{1}{\xi}\mathrm{d}\xi, \quad J_2 = \frac{1}{2}\int_\omega^1 E_2 e_2 \frac{1}{\xi}\mathrm{d}\xi$$

$$J_0 = -\int_\omega^1 E_o e_o \frac{1}{\xi}\mathrm{d}\xi, \quad J_4 = -\frac{1}{2}\int_\omega^1 E_4 e_4 \frac{1}{\xi}\mathrm{d}\xi, \qquad (5.17)$$

and

$$\beta Y = (1-\mu)I_{01}e_o'(1), \quad I_{01} = \int_\omega^1 (E_o \log\xi)\frac{1}{\xi}\mathrm{d}\xi. \qquad (5.18)$$

The denominators of Nu_o and Nu_i, equation (5.13), are also obtained as

$$DENO = (1-\mu)I_{01}e_o'(1) + J_0 + J_2 + J_4$$
$$DENI = DENO - (1-\mu)I_{00}e_o'(1)\log\omega. \qquad (5.19)$$

Upon substitution of equation (5.19) into equation (5.13) the Nusselt numbers then become

$$Nu_o = (DE)_1\left[\frac{\mu e_o'(1)}{DENO}\right]I_{00},$$

$$Nu_i = (DE)_\omega\left[\frac{I_{00}-\mu e_o'(1)}{DENI}\right]I_{00}. \quad (5.20)$$

The results for the integrals J_{01}, J_0, J_2, J_4 are given in the Appendix.

6. NUSSELT NUMBERS ON THE OUTER AND INNER WALLS

From equations (A.3), (A.5) and (A.7) of the Appendix, the last three terms of $DENO$ in equation (5.19) reduce to

$$48(J_0+J_2+J_4) = K_0+K_1u+K_2u^2+K_3u^3+K_4m^4\log\omega \quad (6.1)$$

where

$$K_0 = \frac{11}{8}(1-\omega^8)\left(1+\frac{m^{16}}{\omega^8}\right) - 3\frac{(1-\omega^2)}{(1+\omega^2)}(1+\omega^4)\left(1+\frac{m^8}{\omega^4}\right)m^4 - 24\frac{(1-\omega^2)}{(1+\omega^2)}\left(1-\frac{m^4}{\omega^2}\right)^2 m^4\omega^2 - \frac{32}{3}\frac{(1-\omega^6)}{(1+\omega^2)^2}\left(1+\frac{m^8}{\omega^2}\right)m^4 + 22\frac{(1-\omega^2)}{(1+\omega^2)}m^8 - \frac{32}{3}\frac{(1-\omega^6)}{(1+\omega^2)^3}\left(1-\frac{m^4}{\omega^4}\right)^2 m^4\omega^4 - 32\frac{(1-\omega^2)}{(1+\omega^2)^3}\left(1-\frac{m^4}{\omega^4}\right)(1-m^4)m^4\omega^4 + \frac{1}{3}\frac{(1-\omega^2)}{(1+\omega^2)}\frac{(1-\omega^2)^2}{(1+\omega^4)}m^8,$$

$$K_1 = -\frac{19}{6}(1-\omega^6)\left(1-\frac{m^{12}}{\omega^6}\right)+\frac{39}{2}(1-\omega^2)\left(1-\frac{m^4}{\omega^2}\right)m^4 - \frac{27}{16}(1+\omega^2)(1-\omega^4)\left(1+\frac{m^4}{\omega^2}\right)\left(1-\frac{m^8}{\omega^4}\right)$$

$$K_2 = \frac{15}{8}(1-\omega^4)\left(1+\frac{m^8}{\omega^4}\right)+\frac{9}{2}(1-\omega^2)\left(1+\frac{m^4}{\omega^2}\right)^2,$$

$$K_3 = -3(1-\omega^2)\left(1+\frac{m^4}{\omega^2}\right)^2\bigg/\left(1-\frac{m^4}{\omega^2}\right),$$

$$K_4 = -24\frac{\omega^4}{(1+\omega^2)^2}\left(1-\frac{m^4}{\omega^2}\right)^2. \quad (6.2)$$

Now since all the terms involved in equation (5.20) are expressed by equations (2.14), (4.9), (A.1) and (6.1), the equations (5.20) form analytic closed expressions for the outer and inner Nusselt numbers.

The values of the parameter μ depend on the desired heating and cooling combinations on the walls. For three special cases, insulated inner wall, insulated outer wall and equal wall temperatures, its values were given in equation (4.12). For calculation purposes, two additional special values were selected as

$$\lambda = 2: \quad \mu = 2\frac{I_{00}}{e_o'(1)}. \quad \left\{\frac{U_i}{U_o} = -\frac{1}{2}\right\}$$

$$\lambda = -1: \quad \mu = -\frac{I_{00}}{e_o'(1)} \quad \left\{\frac{U_i}{U_o} = -2\right\} \quad (6.3)$$

which represent heating and cooling conditions where the wall heat fluxes are equidirectional (of opposite sign).

With these five different values of μ, the numerical values of the outer and inner Nusselt numbers were calculated over the range of the values of m and ω. The results were tabulated* and plotted in Figs. 2–6 for four values of the ellipticity, m, and also for $m = \omega$, where the inner pipe reduces to a flat core.

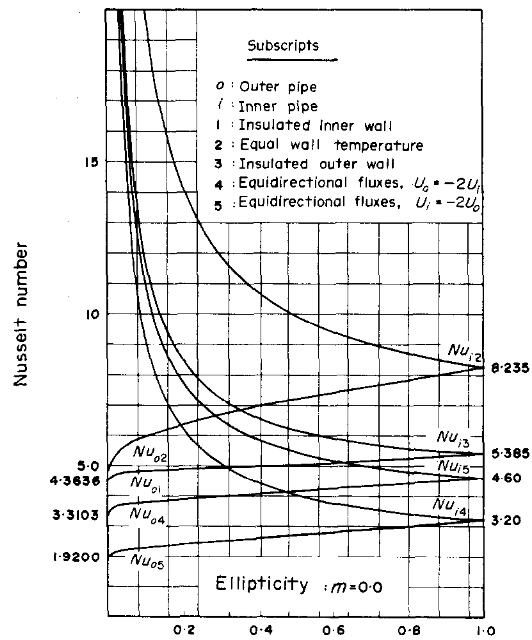

FIG. 2. Variation of Nusselt number with core size for circular annular pipes.

The values of ω numerically start from 0·00005 since the inner Nusselt numbers approach infinity as ω approaches zero. As ω approaches unity, the expressions for outer and inner Nusselt numbers become indeterminate forms of higher order in terms of $(1-\omega^2)$. For this reason, the limiting values of the Nusselt numbers for ω equal unity could not be obtained by numerical computation. However, these values can be and were obtained by graphical extrapolation and are shown on the plots.

*See footnote on p. 1488.

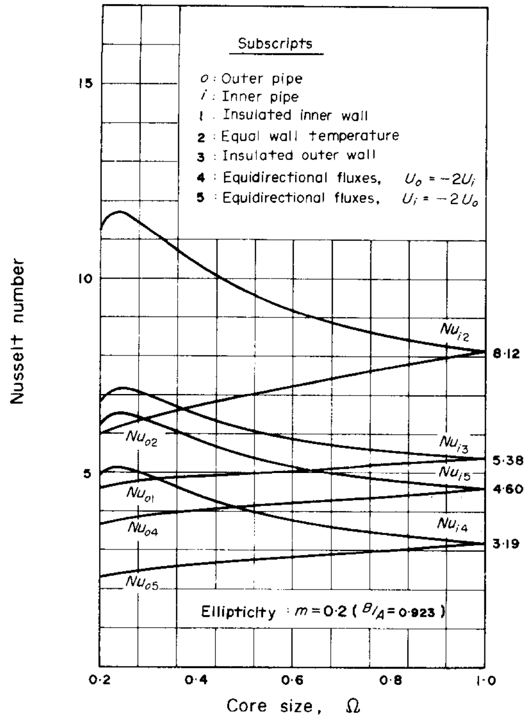

Fig. 3. Variation of Nusselt number with core size for ellipticity of 0·2.

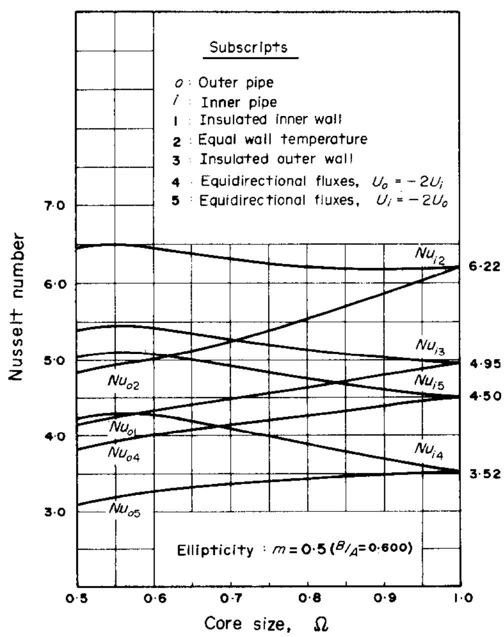

Fig. 4. Variation of Nusselt number with core size for ellipticity of 0·5.

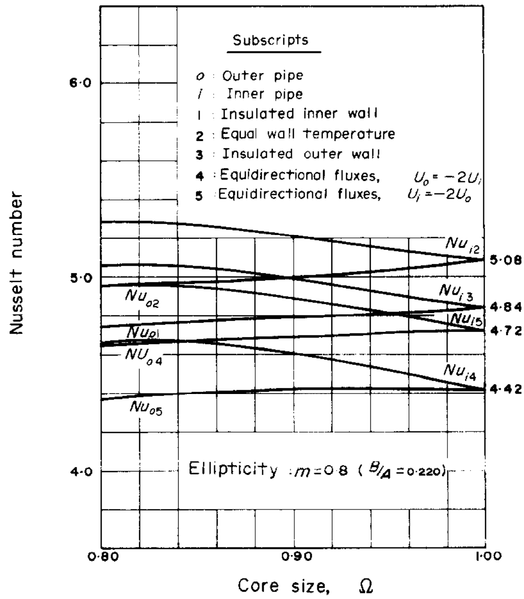

Fig. 5. Variation of Nusselt number with core size for ellipticity of 0·8.

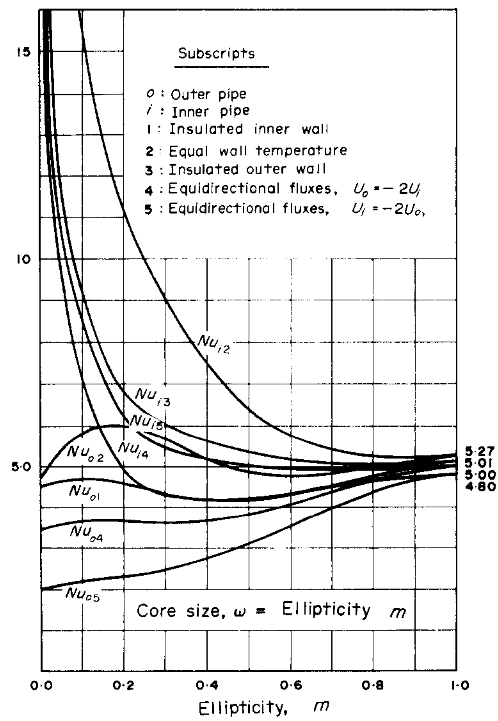

Fig. 6. Variation of Nusselt number with ellipticity for a pipe with a flat core.

Table 1
Ellipticity = 0·00000; Outer pipe flatness (B/A) = 1·00000

Core size	Insulated inner wall	Insulated outer wall	Equal wall temperature		$\frac{\text{Outer flux}}{\text{Inner flux}} = -2$		$\frac{\text{Inner flux}}{\text{Outer flux}} = -2$	
Omega	NUSSO Nu	NUSSI Nu	NUSSO Nu	NUSSI Nu	NUSSO Nu	NUSSI Nu	NUSSO Nu	NUSSI Nu
0·00005	4·513587	4492·779343	4·748549	7467·305433	3·433289	4227·938197	1·998385	4423·506422
0·02000	4·734244	32·705116	5·404413	55·319672	3·633727	28·015864	2·140790	31·391549
0·04000	4·778025	20·509247	5·602903	34·772268	3·685243	17·039185	2·185623	19·515647
0·06000	4·803225	15·933494	5·746783	27·026163	3·720719	12·936929	2·219852	15·061335
0·08000	4·820699	13·468057	5·864669	22·832515	3·749275	10·727405	2·249432	12·659491
0·10000	4·834212	11·905784	5·966896	20·161660	3·774073	9·325270	2·276421	11·135428
0·12000	4·845511	10·819024	6·058514	18·293714	3·796591	8·347368	2·301777	10·073345
0·14000	4·855541	10·015642	6·142415	16·904935	3·817634	7·621986	2·326021	9·286541
0·16000	4·864854	9·395756	6·220420	15·826880	3·837680	7·060022	2·349468	8·678000
0·18000	4·873796	8·901996	6·293761	14·962723	3·857036	6·610373	2·372316	8·192013
0·20000	4·882587	8·498921	6·363308	14·252605	3·875905	6·241502	2·394701	7·794174
0·22000	4·891374	8·163384	6·429706	13·657401	3·894425	5·932828	2·416716	7·462014
0·24000	4·900254	7·879590	6·493442	13·150394	3·912695	5·670313	2·438430	7·180200
0·26000	4·909293	7·636367	6·554895	12·712675	3·930787	5·444035	2·459891	6·937889
0·28000	4·918534	7·425580	6·614367	12·330461	3·948750	5·246767	2·481137	6·727184
0·30000	4·928005	7·241154	6·672101	11·993456	3·966623	5·073114	2·502197	6·542188
0·32000	4·937722	7·078457	6·728297	11·693801	3·984434	4·918961	2·523093	6·378404
0·34000	4·947694	6·933891	6·783121	11·425388	4·002205	4·781113	2·543843	6·232337
0·36000	4·957926	6·804620	6·836713	11·183393	4·019950	4·657046	2·564460	6·101231
0·38000	4·968418	6·688370	6·889189	10·963953	4·037681	4·544740	2·584956	5·982880
0·40000	4·979165	6·583303	6·940652	10·763935	4·055406	4·442559	2·605340	5·875494
0·42000	4·990164	6·487912	6·991188	10·580772	4·073133	4·349160	2·625620	5·777610
0·44000	5·001408	6·400949	7·040874	10·412334	4·090865	4·263431	2·645801	5·688012
0·46000	5·012890	6·321374	7·089775	10·256845	4·108605	4·184445	2·665890	5·605690
0·48000	5·024601	6·248310	7·137950	10·112805	4·126355	4·111419	2·685890	5·529789
0·50000	5·036533	6·181015	7·185449	9·978945	4·144118	4·043689	2·705806	5·459587
0·52000	5·048678	6·118854	7·232320	9·854175	4·161892	3·980687	2·725640	5·394465
0·54000	5·061027	6·061284	7·278603	9·737562	4·179679	3·921926	2·745396	5·333892
0·56000	5·073572	6·007834	7·324333	9·628297	4·197479	3·866983	2·765076	5·277410
0·58000	5·086304	5·958094	7·369546	9·525676	4·215289	3·815492	2·784683	5·224619
0·60000	5·099215	5·911709	7·414269	9·429083	4·233111	3·767130	2·804218	5·175170
0·62000	5·112297	5·868366	7·458531	9·337980	4·250943	3·721618	2·823683	5·128758
0·64000	5·125542	5·827790	7·502356	9·251889	4·268783	3·678705	2·843080	5·085114
0·66000	5·138944	5·789736	7·545766	9·170390	4·286632	3·638172	2·862410	5·043998
0·68000	5·152493	5·753990	7·588782	9·093105	4·304487	3·599823	2·881674	5·005201
0·70000	5·166184	5·720359	7·631423	9·019702	4·322348	3·563484	2·900874	4·968531
0·72000	5·180010	5·688670	7·673706	8·949881	4·340213	3·528998	2·920012	4·933822
0·74000	5·193965	5·658771	7·715647	8·883373	4·358081	3·496226	2·939087	4·900921
0·76000	5·208042	5·630522	7·757261	8·819936	4·375951	3·465041	2·958101	4·869693
0·78000	5·222236	5·603800	7·798562	8·759350	4·393823	3·435329	2·977055	4·840014
0·80000	5·236540	5·578491	7·839562	8·701419	4·411693	3·406987	2·995950	4·811774
0·82000	5·250950	5·554493	7·880274	8·645962	4·429563	3·379921	3·014787	4·784871
0·84000	5·265460	5·531714	7·920708	8·592815	4·447430	3·354045	3·033566	4·759215
0·86000	5·280066	5·510070	7·960874	8·541830	4·465294	3·329283	3·052288	4·734721
0·88000	5·294763	5·489482	8·000783	8·492869	4·483153	3·305562	3·070955	4·711314
0·90000	5·309546	5·469882	8·040442	8·445805	4·501007	3·282818	3·089566	4·688924
0·92000	5·324412	5·451206	8·079865	8·400536	4·518854	3·260991	3·108123	4·667487
0·94000	5·339358	5·433396	8·119067	8·356954	4·536696	3·240026	3·126625	4·646946
0·96000	5·354393	5·416410	8·158111	8·315024	4·554534	3·219866	3·145068	4·627250
0·98000	5·367086	5·397791	8·185761	8·263302	4·571489	3·201312	3·164298	4·607473

Table 2
Ellipticity = 0·70000; Outer pipe flatness $(B/A) = 0·34228$

Core size	Insulated inner wall	Insulated outer wall	Equal wall temperature		$\frac{\text{Outer flux}}{\text{Inner flux}} = -2$		$\frac{\text{Inner flux}}{\text{Outer flux}} = -2$	
Omega	NUSSO Nu	NUSSI Nu	NUSSO Nu	NUSSI Nu	NUSSO Nu	NUSSI Nu	NUSSO Nu	NUSSI Nu
0·70005	4·491059	5·089366	4·794629	5·449662	4·346933	4·518858	3·965183	4·933647
0·72000	4·510064	5·102802	4·808155	5·455865	4·368311	4·541335	3·991911	4·949810
0·74000	4·526764	5·100856	4·821397	5·448053	4·386597	4·546712	4·013751	4·950031
0·76000	4·542452	5·088991	4·835648	5·431993	4·403064	4·539931	4·031899	4·939641
0·78000	4·557786	5·070258	4·851517	5·410816	4·418384	4·523867	4·047041	4·921649
0·80000	4·573173	5·046750	4·869350	5·386615	4·432986	4·500580	4·059650	4·898146
0·82000	4·588877	5·020005	4·889352	5·360887	4·447165	4·471644	4·070090	4·870681
0·84000	4·605075	4·991183	4·911643	5·334729	4·461127	4·438306	4·078650	4·840441
0·86000	4·621881	4·961177	4·936284	5·308954	4·475022	4·401573	4·085569	4·808346
0·88000	4·639366	4·930669	4·963294	5·284163	4·488955	4·362260	4·091052	4·775118
0·90000	4·657567	4·900188	4·992665	5·260795	4·502999	4·321037	4·095277	4·741318
0·92000	4·676500	4·870138	5·024368	5·239167	4·517206	4·278455	4·098400	4·707388
0·94000	4·696161	4·840824	5·058356	5·219499	4·531609	4·234966	4·100562	4·673669
0·96000	4·716540	4·812483	5·094594	5·201956	4·546229	4·190937	4·101880	4·640431
0·98000	4·739222	4·786901	5·136788	5·190438	4·561822	4·145462	4·101264	4·608624

Table 3
Ellipticity = 0·80000; Outer pipe flatness $(B/A) = 0·21951$

Core size	Insulated inner wall	Insulated outer wall	Equal wall temperature		$\frac{\text{Outer flux}}{\text{Inner flux}} = -2$		$\frac{\text{Inner flux}}{\text{Outer flux}} = -2$	
Omega	NUSSO Nu	NUSSI Nu	NUSSO Nu	NUSSI Nu	NUSSO Nu	NUSSI Nu	NUSSO Nu	NUSSI Nu
0·80005	4·745782	5·054832	4·956741	5·283583	4·645150	4·658596	4·367328	4·949586
0·82000	4·756303	5·058862	4·963491	5·283147	4·657370	4·669332	4·383815	4·955511
0·84000	4·765008	5·050822	4·970147	5·271983	4·667047	4·665929	4·395927	4·948766
0·86000	4·773000	5·034932	4·977784	5·254418	4·675288	4·652390	4·404769	4·933518
0·88000	4·780833	5·013539	4·986899	5·232793	4·682672	4·631047	4·410971	4·912112
0·90000	4·788851	4·988266	4·997776	5·208691	4·689569	4·603580	4·414977	4·886190
0·92000	4·797277	4·960316	5·010583	5·183257	4·696234	4·571286	4·417125	4·856980
0·94000	4·806261	4·930610	5·025416	5·157342	4·702848	4·535204	4·417689	4·825433
0·96000	4·815904	4·899866	5·042326	5·131592	4·709542	4·496177	4·416892	4·792297
0·98000	4·826421	4·868799	5·061670	5·106839	4·716483	4·454772	4·414797	4·758241

7. SPECIAL CASE OF CIRCULAR PIPES

For circular concentric pipes, $m = 0$, the terms involved in equation (5.20) reduce to

$$u = -\frac{(1-\omega^2)}{\log\omega}, \quad I_{00} = \tfrac{1}{4}(1-\omega^2)(1+\omega^2-u),$$

$$e_o'(1) = \tfrac{1}{16}[4-4u-3(1+\omega^2)u+4u^2],$$

$$I_{01} = -\tfrac{1}{16}[3(1-\omega^4)+4(1-\omega^2)\omega^2-4(1-\omega^2)u + 4\omega^4\log\omega],$$

$$48(J_0+J_2+J_4) = \tfrac{11}{8}(1-\omega^8)-\tfrac{19}{6}(1-\omega^6)u - \tfrac{27}{6}(1+\omega^2)(1-\omega^4)u+\tfrac{51}{8}(1-\omega^4)u^2-3(1-\omega^2)u^3,$$

$$(DE)_1 = 2(1-\omega), \quad (DE)_\omega = 2\frac{(1-\omega)}{\omega}. \quad (7.1)$$

For a simple circular pipe, $\omega = 0$, the terms in equation (7.1) further simplify, giving

$$I_{00} = \tfrac{1}{4}, \quad e_o'(1) = \tfrac{1}{4}, \quad I_{01} = -\tfrac{3}{16},$$

$$J_0+J_2+J_4 = \frac{11}{384}, \quad (DE)_1 = 2, \quad (DE)_\omega \approx \frac{2}{\omega} \quad (7.2)$$

where $(DE)_\omega \approx$ means that this term approaches its limits as $(2/\omega)$. Substitution of equation (7.2) into equation (5.20) gives the outer Nusselt number for a circular pipe with a very thin central wire as

$$Nu_o = \frac{48\mu}{11-18(1-\mu)}. \qquad (7.3)$$

The inner Nusselt number for this case is

$$Nu_i = \infty \text{ when } \mu \neq 1; \quad Nu_i = 0 \text{ when } \mu = 1. \qquad (7.4)$$

It is to be noted that for $\mu = 1$, the value of the inner Nusselt number as a function of ω has a discontinuity near the origin. That is when ω is slightly greater than zero, the inner Nusselt number becomes very large.

When the selected values of μ are used in equation (7.3), the Nusselt numbers on the outer wall are obtained as:

For insulated central wire:

$$\mu = \frac{I_{00}}{e_0'(1)} = 1, \quad Nu_o = \frac{48}{11} = 4{\cdot}3636;$$

For equal wall temperatures:

$$\mu = 1, \quad Nu_o = \frac{48}{11} = 4{\cdot}3636;$$

For insulated outer wall:

$$\mu = 0, \quad Nu_o = 0;$$

For equidirectional heat fluxes and $U_i = -\frac{1}{2}U_o$:

$$\mu = 2\frac{I_{00}}{e_0'(1)} = 2, \quad Nu_o = \frac{96}{29} = 3{\cdot}3103;$$

For equidirectional heat fluxes and $U_i = -2U_o$:

$$\mu = -\frac{I_{00}}{e_0'(1)} = -1, \quad Nu_o = \frac{48}{25} = 1{\cdot}9200. \qquad (7.5)$$

For the last three cases the central wire acts as a line heat source or sink of infinite strength.

For circular concentric pipes, the numerical values of the Nusselt numbers on the outer and inner walls for five wall heating conditions are as given in Table 1 and plotted in Fig. 2. These values agree with the data given in [2] to the extent of the number of significant digits as given in this reference. When ω approaches unity, the Nusselt numbers approach to those values for a flow between two parallel plates which are heated or cooled according to the wall conditions considered. The first two of these limiting cases, as shown in Fig. 2, agree with those given in [1]. The other two are for equidirectional heat fluxes on the walls, as given by equation (6.3).

8. CONCLUSIONS

Convective heat transfer in the annulus of two confocal elliptic pipes is analyzed and results are presented in analytical closed form.

For the special case of two concentric circular pipes and when ω equals unity, the results are for the flow between two parallel plates. Since when $\omega = 1$ the walls are interchangeable, the values of Nu_o and Nu_i for the same heating conditions on either wall must yield the same value. This property has been verified.

For the case of elliptic annular pipes this property prevails. However, the common value of the Nusselt number for $\omega = 1$ changes with different eccentricities. The reason for this is that only in the circular concentric pipes, does the flow approach the limiting case of flow between parallel plates. Whereas, in the case of confocal elliptic pipes, the local Nusselt number and the Nusselt number, as defined here for elliptic pipes, are not one and the same and they do not approach that of parallel plates.

It is seen from Figs. 3–6, that some interesting optimum heat-transfer properties exist. For some engineering applications, these optima may prove to be useful.

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APPENDIX

$$I_{01} = A_{01} + B_{01}u + D_{01}\log\omega \qquad (A.1)$$

where

$$A_{01} = -\frac{3}{16}(1-\omega^4)\left(1-\frac{m^8}{\omega^4}\right) - \frac{1}{4}(1-\omega^2)\left(1-\frac{m^4}{\omega^2}\right)\left(1+\frac{m^4}{\omega^4}\right)\omega^2$$

$$B_{01} = \frac{1}{4}(1-\omega^2)\left(1+\frac{m^4}{\omega^2}\right)$$

$$D_{01} = -\frac{1}{4}\left(1-\frac{m^8}{\omega^8}\right)\omega^4 - \left[\frac{(1-\omega^2)}{(1+\omega^2)}\right]m^4. \qquad (A.2)$$

$$J_0 = \tfrac{1}{16}(U_0 + V_0 u + W_0 u^2 + Y_0 u^3 + Z_0 m^4 \log\omega \qquad (A.3)$$

where

$$U_0 = \frac{11}{24}(1-\omega^8)\left(1+\frac{m^{16}}{\omega^8}\right) + \frac{2}{3}\frac{(1-\omega^2)}{(1+\omega^2)}(1+\omega^4)\left(1+\frac{m^8}{\omega^4}\right)m^4 - 8\frac{(1-\omega^2)}{(1+\omega^2)}\left(1-\frac{m^4}{\omega^2}\right)^2 m^4\omega^2$$

$$V_0 = -\frac{19}{18}(1-\omega^6)\left(1-\frac{m^{12}}{\omega^6}\right) + \frac{7}{2}(1-\omega^2)\left(1-\frac{m^4}{\omega^2}\right)m^4 - \frac{9}{16}(1+\omega^2)\left(1+\frac{m^4}{\omega^2}\right)(1-\omega^4)\left(1-\frac{m^8}{\omega^4}\right)$$

$$W_0 = \frac{5}{8}(1-\omega^4)\left(1+\frac{m^8}{\omega^4}\right)+\frac{3}{2}(1-\omega^4)\left(1+\frac{m^4}{\omega^2}\right)^2$$

$$Y_0 = -(1-\omega^2)\left(1+\frac{m^4}{\omega^2}\right)^2 \Big/ \left(1-\frac{m^4}{\omega^2}\right)$$

$$Z_0 = 2m^4+8(1-\omega^2)^2\frac{m^4}{\omega^2}+32\frac{m^4\omega^2}{(1+\omega^2)^2} + \frac{8}{3}(1-\omega^2)^2\left(1-\frac{m^4}{\omega^2}\right)^2. \quad (A.4)$$

$$J_2 = \frac{1}{6}\frac{m^4}{(1+\omega^2)}(U_2+V_2u+Z_2\log\omega) \quad (A.5)$$

where

$$U_2 = -\frac{5}{8}(1-\omega^2)(1+\omega^4)\left(1+\frac{m^8}{\omega^4}\right)-\frac{4}{3}\frac{(1-\omega^6)}{(1+\omega^2)}\left(1+\frac{m^8}{\omega^2}\right) + 4(1-\omega^2)m^4-\frac{4}{3}\frac{(1-\omega^6)}{(1+\omega^2)^2}\left(1-\frac{m^4}{\omega^4}\right)\omega^4 - 4\frac{(1-\omega^2)}{(1+\omega^2)^2}\left(1-\frac{m^4}{\omega^4}\right)(1-m^4)\omega^4$$

$$V_2 = \frac{9}{8}(1-\omega^4)\left(1-\frac{m^4}{\omega^2}\right)$$

$$Z_2 = -3\frac{(1+m^4\omega^2)}{(1+\omega^2)}\left(\omega^4+\frac{m^4}{\omega^2}\right) - (1-\omega^2)(1-\omega^4)\left(1-\frac{m^4}{\omega^2}\right)^2+2\frac{m^4\omega^2}{(1+\omega^2)}. \quad (A.6)$$

$$J_4 = \tfrac{1}{24}m^8(U_4+Z_4\log\omega) \quad (A.7)$$

where

$$U_4 = -5\frac{(1-\omega^2)}{(1+\omega^2)}+\frac{1}{6}\frac{(1-\omega^2)(1-\omega^2)^2}{(1+\omega^2)(1+\omega^4)}$$

$$Z_4 = -3-8\frac{\omega^2}{(1+\omega^2)^2}. \quad (A.8)$$

CONVECTION THERMIQUE LAMINAIRE ET PERMANENTE ENTRE DEUX TUBES A SECTIONS ELLIPTIQUES ET HOMOFOCALES AVEC GRADIENT LONGITUDINAL UNIFORME DE TEMPERATURE PARIETALE

Résumé—On présente la solution mathématique de la convection thermique laminaire et permanente entre deux tubes à sections elliptiques et homofocales dans différentes conditions de chauffage, en utilisant les équations de Navier–Stokes et d'énergie. On montre que le problème est conditionné par un nombre caractéristique qui est le produit du gradient adimensionnel, longitudinal, uniforme de la température pariétale par le nombre de Péclet. On obtient la valeur du nombre de Nusselt pour différentes formes de la section de passage.

KONVEKTIVE WÄRMEÜBERTRAGUNG BEI STATIONÄRER LAMINARER STRÖMUNG ZWISCHEN ZWEI KONFOKALEN ELLIPTISCHEN ROHREN MIT GLEICHFÖRMIGEN TEMPERATURGRADIENTEN IN LÄNGSRICHTUNG

Zusammenfassung—Die konvektive, stationäre, laminare Wärmeübertragung zwischen zwei konfokalen elliptischen Rohren mit gleichförmigen Temperaturgradienten in Längsrichtung wird unter verschiedenen Heizbedingungen in einer analytisch geschlossenen Form unter Anwendung der Navier–Stokes- und der Energiegleichung dargestellt. Es wird gezeigt, daß eine charakteristische Zahl, das Produkt des dimensionslosen, gleichförmigen Wandtemperaturgradienten und der Péclet-Zahl, das Problem beschreibt. Die Nusselt-Zahlen für verschiedene Werte der Elliptizität und für verschiedene Abmessungen werden angegeben.

КОНВЕКТИВНЫЙ ТЕПЛООБМЕН В СТАЦИОНАРНОМ ЛАМИНАРНОМ ПОТОКЕ МЕЖДУ ДВУМЯ КОАКСИАЛЬНЫМИ ЭЛЛИПТИЧЕСКИМИ ТРУБКАМИ С ПОСТОЯННЫМ ПРОДОЛЬНЫМ ГРАДИЕНТОМ ТЕМПЕРАТУРЫ СТЕНКИ

Аннотация — В статье приводится решение задачи конвективного теплообмена в ламинарном потоке между двумя коаксиальными эллиптическими трубками с постоянным продольным градиентом температуры на стенке при различных условиях нагрева, полученное в замкнутом виде с использованием точных решений уравнения Навье–Стокса и уравнения сохранения энергии. Показано, что решение задачи зависит от одного характеристического числа — произведения постоянного безразмерного продольного градиента температуры на стенке и числа Пекле. Получены значения числа Нуссельта для нескольких значений эллиптичности и размеров внутренней трубки.